

# Probing Quantum Phase Transition in Macroscopic Qubit Array via circuit QED architecture

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(Dated: February 1, 2008)

We demonstrate a universal physical mechanism to probe the macroscopic quantum phase transition based on circuit QED architecture. We found that, with certain parameters, the Josephson junction qubit array behaves as an antiferromagnetic Ising model in transverse field and the coupled transmission line resonator serves as a bosonic quantum probe. Our investigation shows that, at the critical point, the drastic broadening of the spectrum of the probe indicates the quantum phase transition.

PACS numbers: 74.81.Fa, 42.50.Pq, 75.10.Pq, 73.43.Nq

*Introduction*— Non-analyticity in ground state energy of a quantum many body system at critical point is referred to as quantum phase transition (QPT) [1], which is essentially caused by quantum fluctuations even at zero temperature. Some recent investigations have discovered that [2, 3, 4] the critical behavior of a system with QPT can enhance quantum decoherence of its coupled external system. Here, the enhanced decoherence is displayed by a sensitive decay of the Loschmidt echo (LE) [5] at critical point and possesses some universality in the ordered domain [4]. These discoveries enlighten us to propose a scheme to probe the intrinsic QPT phenomena of a system by detecting the exotic spectral structure of its coupled system.

On the other hand, as a macroscopic QPT phenomenon, the superfluid-Mott insulator transition has been demonstrated in a macroscopic quantum system – the atomic Bose-Einstein condensate in an optical lattice [6]. It is natural to extend the research on QPT to some other macroscopic quantum systems, such as the superconducting Josephson junction (JJ) array system. For the generic JJ array, much effort has been devoted to the superfluid-Mott insulator transition, see Ref.[2] and references therein. In "qubit" regime, this system has been studied for some other purposes [7, 8, 9], e.g., quantum state transfer.

In this paper, we investigate the macroscopic QPT of an Ising chain in transverse field (ITF) implemented with JJ qubit array for the first time. We present and study a physical mechanism to probe its QPT with a coupled on-chip superconducting transmission line resonator (TLR) [10]. We find that, when the QPT occurs in the JJ qubit array, the spectrum of the TLR is significantly changed from discrete-peak structure into almost white noise spectrum. This drastic broadening of the spectrum serves as a witness of QPT. We also discuss the universality of QPT exhibited in the spectrum structure.

*QPT model based on JJ qubit array and 1D TLR* – We consider a quantum network including  $N$  Cooper pair boxes (CPBs) (see Fig.(1)). Each CPB is a dc-SQUID formed by a superconducting island connected to

two Josephson junctions. The effective Josephson tunnelling energy can be tuned by the magnetic flux  $\Phi_x$  threading the dcSQUID. With proper bias voltage, the CPB behaves as a qubit [11] and then JJ qubit array becomes an engineered "spin" chain with  $N/2$  -spins. When the coupling capacitance  $C_m$  between two CPBs is much smaller than the total one  $C_\Sigma$  (e.g, in ref. [12],  $C_m/C_\Sigma \approx 0.05$ ), the terms  $\sim o(C_m/C_\Sigma)$  in Hamiltonian can be neglected and we only consider the nearest neighbor interaction in this "spin" chain. Then the JJ qubit array can be described by a 1D ITF model with the effective Hamiltonian

$$H_0 = \hat{h}(\lambda) \equiv B \sum_{\alpha=1}^N (\lambda \sigma_x^{(\alpha)} + \sigma_z^{(\alpha)} \sigma_z^{(\alpha+1)}), \quad (1)$$

where  $\lambda = B_x/B$  and  $B = e^2 C_m / C_\Sigma^2$  characterizes the Coulomb interaction between nearest neighbors. The Josephson energy of CPB  $B_x = E_J \cos(\Phi_x / \Phi_0) / 2$  with  $E_J$  the Josephson energy of single junction and  $\Phi_0 = h/2e$  the flux quantum. For simplicity, all qubits are assumed to be identical and biased at the degenerate point. The quasi-spin operators  $\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$ ,  $\sigma_x = -|0\rangle\langle 1| - |1\rangle\langle 0|$  are defined in terms of the charge eigenstates  $|0\rangle$  and  $|1\rangle$ .  $|0\rangle$  and  $|1\rangle$  denote 0 and 1 excess Cooper pair on the island respectively. A most recent ex-

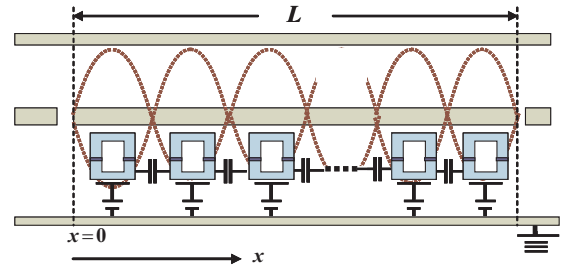


FIG. 1: (Color on line) The schematics of our setup. A capacitively coupled Josephson junction qubit array is placed in a 1D TLR. Each qubit is coupled with the quantized magnetic field of the TLR.

periment has demonstrated the possibility to implement a four-JJ-qubit Ising array [13].

In our setup, as a quantum probe, a 1D TLR of length  $L$  is placed in parallel with this JJ qubit array (see Fig.(1)) away from a distance  $d$ . Each CPB sits at the antinodes  $x = (2n + 1)L/2N$  ( $n = 0, \dots, N - 1$ ) of the magnetic field induced by the current  $J$  in the TLR [10]. Since  $J$  vanishes at the end of the TLR, the London equation provides the boundary condition for the electromagnetic field of this on-chip resonator. Thus, the electric field vanishes at those antinodes and the qubits are only coupled with the magnetic component. The magnetic flux threading each dcSQUID is  $\phi_x = \eta(a + a^\dagger)$  with  $\eta = (S/d)(\hbar l\omega/L)^{1/2}$  where  $l$  is the inductance per unit length and  $S$  is the enclosed area of the dcSQUID. Here, we have assumed only a single mode of magnetic field with frequency  $\omega$  is coupled with JJ qubit array [14] and  $a$  ( $a^\dagger$ ) is its annihilation (creation) operator. Usually,  $\eta$  is small enough for the harmonic approximation [2, 15]  $\cos \phi_x \approx 1 - \phi_x^2$  and the Hamiltonian  $H = H_0 + H_F$  takes a spin-boson form

$$H_F = \hbar\omega a^\dagger a - g \sum_{\alpha} (a^\dagger a + aa^\dagger) \sigma_x^{(\alpha)}, \quad (2)$$

with the coupling coefficient  $g = \eta E_J$ . Here, we have already invoked the rotation wave approximation to neglect the high frequency terms proportional to  $a^\dagger a^\dagger$  and  $a^2$  under the condition  $\omega \gg B_x, B$ . This approximation condition can be satisfied with accessible parameters in current experiments. For example, if we take  $C_\Sigma \sim 600$  aF and  $C_\Sigma \sim 30$  aF,  $L \sim 1$  cm,  $S \sim 10 \mu\text{m}^2$ ,  $d \sim 1 \mu\text{m}$  and  $N = 500$ , then  $B = 1.6$  GHz,  $E_J = 13$  GHz,  $\omega \sim 120$  GHz,  $\eta \sim 0.01$  [16].

*Pseudo-spin representation for paired excitation spectrum* - By introducing the Jordan-Wigner transformation  $\sigma_z^{(\alpha)} = \prod_{\beta < \alpha} (2c_\beta^\dagger c_\beta - 1) (c_\alpha + c_\alpha^\dagger)$  and  $\sigma_x^{(\alpha)} = 1 - 2c_\alpha^\dagger c_\alpha$ ,  $H_0$  can be diagonalized as  $H_0 = \sum_k \varepsilon_k \gamma_k^\dagger \gamma_k$  by the fermionic quasi-particle operator [1, 17, 18]

$$\gamma_k = \sum_{\alpha=1}^N \frac{e^{-ik\alpha}}{\sqrt{N}} (c_\alpha \cos \frac{\theta_k}{2} - ic_\alpha^\dagger \sin \frac{\theta_k}{2}) \quad (3)$$

with dispersion relation  $\varepsilon_k(\lambda) = 2B\sqrt{1 + \lambda^2 - 2\lambda \cos k}$  for  $\tan \theta_k(\lambda) = \sin k / (\lambda - \cos k)$ . The ground state  $|G\rangle$  of  $H_0$  describes the state without any quasi-particle excitation.

With respect to the Fock state  $|n\rangle$  of TLR, the Hamiltonian of the whole system can be decomposed as  $H = \sum_n H_n |n\rangle \langle n|$  where  $H_n = \hat{h}(\lambda_n)$  are defined by Eq.(1) with  $\lambda_n = \lambda - (2n + 1)g/B$  and a constant term  $\hbar n\omega$  has been omitted. For further convenience, we introduce

a set of pseudo-spin operators [19]

$$\begin{aligned} s_{zk} &= \gamma_k^\dagger \gamma_k + \gamma_{-k}^\dagger \gamma_{-k} - 1, \\ s_{xk} &= i(\gamma_{-k} \gamma_k + \gamma_{-k}^\dagger \gamma_k^\dagger), \\ s_{yk} &= \gamma_{-k}^\dagger \gamma_k^\dagger - \gamma_{-k} \gamma_k. \end{aligned} \quad (4)$$

They describe the pairing of quasi-particle excitations by  $\gamma_k$ . With these pseudo-spin operators, each branch Hamiltonian  $H_n$  can be rewritten as  $H_n = \sum_{k>0} H_n^{(k)}$ , where

$$H_n^{(k)} = \varepsilon_{nk}(s_{zk} \cos 2\alpha_{nk} + s_{xk} \sin 2\alpha_{nk}) \quad (5)$$

with  $2\alpha_{nk} = \theta_{nk} - \theta_k$ ,  $\varepsilon_{nk} = \varepsilon_k(\lambda_n)$  and  $\theta_{nk} = \theta_k(\lambda_n)$ .

*Detection of QPT* - We expect to detect the critical behavior of the JJ qubit array by the coherence property of the TLR. To demonstrate the quantum coherence of a single mode electromagnetic field, a natural option is the correlation spectrum function  $S(\omega) = \int dt e^{-i\omega t} S(t)$ , which is the Fourier transformation of the 1st order correlation function of the single mode field

$$S(t) = \langle a^\dagger(t) a(0) \rangle = \sum_n n |c_n|^2 D_{n,n-1}(t) e^{-\Gamma|t|}. \quad (6)$$

Here, the average  $\langle \dots \rangle$  is taken over an initial state  $|\Psi(0)\rangle = |\psi_0\rangle \otimes |G\rangle$  and  $|\psi_0\rangle = \sum_n c_n |n\rangle$  is an arbitrary pure state of the TLR (our discussion here is also valid if  $|\psi_0\rangle$  is an arbitrary mixed state). The decoherence factor  $D_{n,n-1}(t) = \langle G | \exp(iH_n t) \exp(-iH_{n-1} t) | G \rangle$  evaluates the overlap of the wave functions under two different Hamiltonians  $H_n$  and  $H_{n-1}$ . We also phenomenologically introduce the decaying factor  $\exp(-\Gamma|t|)$  in the quasi-mode treatment of dissipation [14]. For strong coupling limit,  $g \gg \Gamma$  and  $\Gamma$  is about 6.3 MHz for the first excitation mode [10].

By carrying out the evaluation of time evolution, we obtain explicitly the spectrum function

$$S(\omega) = \sum_n n |c_n|^2 D_{n,n-1}(\omega), \quad (7)$$

where

$$D_{n,n-1}(\omega) = \sum_{\{(a_k, b_k)\}} \frac{2\Gamma F_{(a_k, b_k)}^{(n)}}{\Gamma^2 + (\omega - \Omega_{\{(a_k, b_k)\}}^{(n)})^2} \quad (8)$$

is the sum of many Lorentzian distributions with the same half width at half maximum (HWHM)  $\Gamma$ , but different central frequencies  $\Omega_{\{(a_k, b_k)\}}^{(n)} = \sum_k (a_k \varepsilon_{nk} + b_k \varepsilon_{n-1, k})$ . The sum in eq.(8) is taken over all the possible configurations of combinations  $\{(a_k, b_k) | a_k, b_k = \pm\}$ , e.g., one possible combination is  $\{(+, -)_1, (+, +)_2, \dots, (-, +)_{N/2}\}$ . Here,  $F_{\{(a_k, b_k)\}}^{(n)} =$

$\prod_k c_{a_k b_k, k}^{(n, n-1)}$  is defined by

$$\begin{aligned} c_{++,k}^{(n, n-1)} &= -\sin \alpha_{nk} \cos \alpha_{n-1k} \sin (\alpha_{n-1k} - \alpha_{nk}), \\ c_{+-,k}^{(n, n-1)} &= \sin \alpha_{nk} \sin \alpha_{n-1k} \cos (\alpha_{n-1k} - \alpha_{nk}), \\ c_{-+,k}^{(n, n-1)} &= \cos \alpha_{nk} \cos \alpha_{n-1k} \cos (\alpha_{n-1k} - \alpha_{nk}), \\ c_{--,k}^{(n, n-1)} &= \cos \alpha_{nk} \sin \alpha_{n-1k} \sin (\alpha_{n-1k} - \alpha_{nk}). \end{aligned} \quad (9)$$

Without considering the decay of quasimodes, those Lorentzian line shapes reduce to delta functions.

The time evolution of the 1st order correlation function  $S(t)$  with  $N = 1000$  is shown in Fig.(2) for different  $\lambda$  with  $|\psi_0\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ . It can be seen that the decay rates for different  $\lambda$  are almost the same except  $\lambda = 1$ . This decay is induced by the dissipation of the quasimodes, which has the same influence for different  $\lambda$ . However, near the critical point, i.e.,  $\lambda \approx 1$ , the decay is drastically enhanced. This means that there exists an extra strong decay mechanism related to QPT.

To illustrate the effect of QPT more clearly, we resort to the behavior of the spectrum function  $S(\omega)$ . The numerical result by FFT is shown in Fig.(3) and (4). In Fig.(3), the left panel is plotted for  $|\psi_0\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$  while the right panel for  $|\psi_0\rangle = |\alpha\rangle$  with  $\alpha = 1$ . It can be seen that, for both of the two initial states, generally there are only one or several Lorentzian peaks centered at discrete frequencies in  $S(\omega)$  while near the phase transition point the spectrum of TLR gets broad and chaotic. As  $N$  increases, this broadened distribution at the critical point becomes more and more smooth and tends to be a white noise spectrum at large  $N$  limit (see Fig.(4)). Thus, the QPT of the JJ qubit array is featured by the intensive broadening in the TLR output spectrum. Hence, from the correlation spectrum of the quantum probe, we can infer the occurrence of QPT.

To investigate the underlying physical mechanism for the behavior described above, we rewrite  $S(\omega)$  as

$$S(\omega) = \sum_{n, ii'} p_{i, ii'}^{(n)} \langle E_i^{(n)} | E_{i'}^{(n-1)} \rangle L(\omega, \Lambda_{ii'}^{(n)}, \Gamma). \quad (10)$$

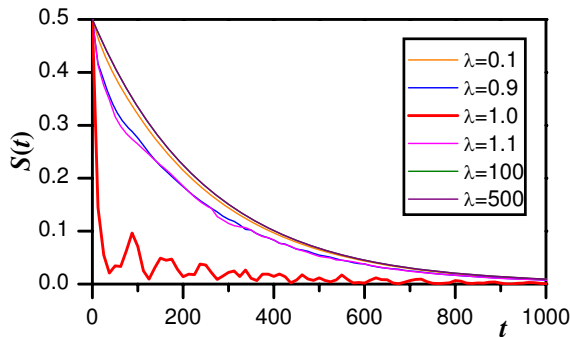


FIG. 2: (Color on line) The 1st order correlation function  $S(t)$  for  $|\psi_0\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$  is plotted with different  $\lambda$ . Here  $N = 1000$  and the time  $t$  is in the unit of  $1/B$ .

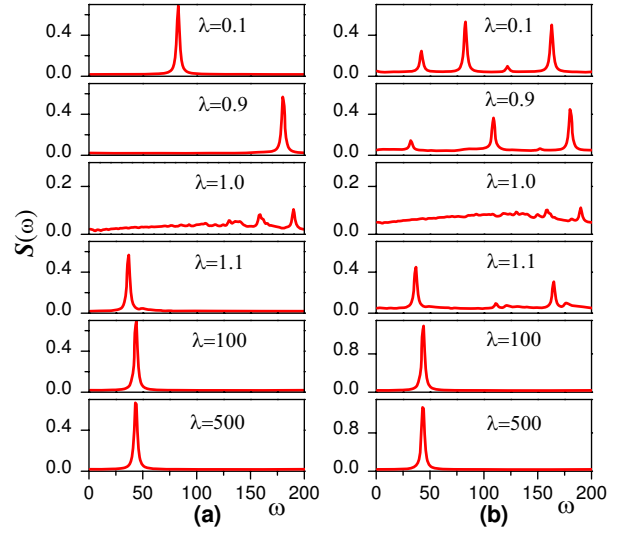


FIG. 3: (Color on line) The spectrum  $S(\omega)$  is shown with different  $\lambda$ . The left panel is plotted for  $|\psi_0\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$  and the right panel is for  $|\psi_0\rangle = |\alpha\rangle$ . Here  $N = 1000$ .

Here  $|E_i^{(n)}\rangle$  is an eigenvector of  $H_n$  with eigenvalue  $E_i^{(n)}$ ,  $p_{i, ii'}^{(n)} = n |c_n|^2 \langle G | E_i^{(n)} \rangle \langle E_{i'}^{(n-1)} | G \rangle$  and

$$L(\omega, \Lambda_{ii'}^{(n)}, \Gamma) = \frac{2\Gamma}{\Gamma^2 + \left(\omega - \Lambda_{ii'}^{(n)}\right)^2} \quad (11)$$

is a Lorentzian function with the HWHM  $\Gamma$  and central frequency  $\Lambda_{ii'}^{(n)} = E_i^{(n)} - E_{i'}^{(n-1)}$ . In some sense,  $S(\omega)$  measures how many different eigenvectors of  $H_{n-1}$  are needed to express one eigenvector of  $H_n$ . The more are necessary, the wider the support of  $S(\omega)$ . This observation provides the intrinsic reason for the widening of the spectrum. Since  $g$  is assumed to be small perturbation one would generally expects that the difference between  $H_n$  and  $H_{n-1}$  is almost negligible and their eigenvectors are very close to each other, that is  $\langle E_i^{(n)} | E_{i'}^{(n-1)} \rangle \approx \delta_{i, i'}$  and

$$S(\omega) \approx L(\omega, 0, \Gamma) \sum_n n |c_n|^2 \quad (12)$$

Then the support of  $S(\omega)$  is very narrow and the corresponding Fourier transformation  $S(t)$  decays very slow.

However, the above analysis is invalid at the critical point. Near the critical point, the property of the QPT system, such as ground state and long range order is significantly influenced by a small perturbation in either of the two competing terms: the Ising interaction and the transverse field. The seemingly very small difference between the two Hamiltonians  $H_n$  and  $H_{n-1}$  actually has drastic impact on the evolutions driven by the two Hamiltonians. This implies that more eigenvectors of  $H_{n-1}$  are needed to reproduce one eigenvector of  $H_n$ . Therefore,

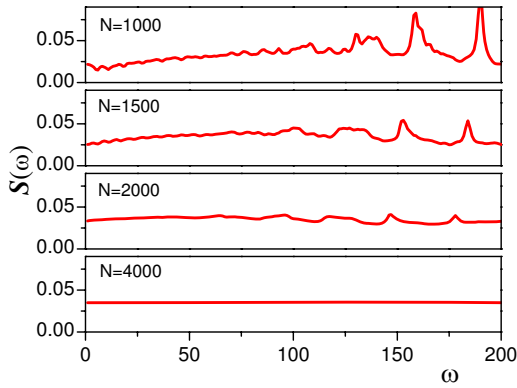


FIG. 4: (Color on line) The spectrum  $S(\omega)$  at the critical point is plotted for different  $N$  with  $|\psi_0\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ .

more Lorentzian shapes have to be included and the support of  $S(\omega)$  becomes much broader. This in turn accelerates the decay of  $S(t)$  and acts as the extra strong decay mechanism related to QPT as we have noticed in Fig.(2).

The mechanism described in the paper can be extended to the case with the two-level atom as probe. The universality similar to Ref [4] is also revealed in Fig.(3). Here, when  $\lambda$  is not large, the location of peaks in  $S(\omega)$  depends on both  $\lambda$  and  $|\psi_0\rangle$ . However, for very large  $\lambda$ , there is only one Lorentzian peak in the spectrum and the location of this peak is independent of  $\lambda$  and  $|\psi_0\rangle$ . In Fig.(2), we can also see that the decay envelope for  $\lambda = 500$  overlaps with  $\lambda = 100$ . This is because the approximation in eq.(12) is rigorously hold only if the JJ qubit array is far away from the critical point. In this case, the spectrum exhibits universal features.

This probe mechanism requires  $g \gg \Gamma$ , which ensures the decoherence related to QPT is far more prominent than that caused by surrounding environment. But  $g$  also should be much smaller than the energy scale of the free qubit array. Otherwise, the QPT nature of the ITF model would be significantly changed.

**Conclusion** – In this paper, with superconducting circuit QED structure, we demonstrate a detection scheme for the macroscopic QPT phenomenon. By examining the coherent output of the coupled TLR, the quantum criticality of the JJ qubit array can be probed. The developing experiments [10, 12, 13] make our scheme to be potentially feasible in the near future. Concerning experimental implementation, we would like to point out only the case of  $B_z = 0$  is discussed here to obtain an analytical result. But due to the unavoidable charge fluctuation in our system, it is hard to set the bias charge to  $1/2$  precisely. Therefore, more realistic consideration reminds us to concern the case of  $B_z \neq 0$ , which is modeled with a

transverse field Ising model also with a longitudinal field. When the longitudinal field is weak enough, this generalized model near critical point can be revealed with a perturbation theory.

This work is funded by NSFC with grant Nos. 90203018, 10474104, 60433050, and NFRPC with Nos. 2001CB309310 and 2005CB724508.

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